16.1 Line Integrals of Scalar Functions

Evaluate $\int_C (x+y)\,ds$ where C is the straight-line segment $x=3t$, $y=(6-3t)$, $z=0$ from $(0,6,0)$ to $(6,0,0)$.

when $x=0$ $t=0$
$x=6$ $t=2$
so $0 \le t \le 2$

$$\int_C f(x,y,z)\,ds = \int_a^b f(g(t),h(t),k(t))\,|v(t)|\,dt$$

$r(t) = 3ti + (6-3t)j + 0k$
$v'(t) = 3i - 3j$
$|v'(t)| = \sqrt{3^2+(-3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2} \rightarrow |v(t)|\,dt$
$ds = |v(t)|\,dt$

so, $\int_C (x+y)\,ds = \int_0^2 (x+y)\,|v(t)|\,dt$

$\int_0^2 (\cancel{3t}+6-\cancel{3t})\cdot 3\sqrt{2}\,dt$

$\int_0^2 18\sqrt{2}\,dt = 18\sqrt{2}\,[t]_0^2 = 36\sqrt{2}$

Integrate $f(x,y,z) = x+\sqrt{y}-z^4$ over the path from $(0,0,0)$ to $(1,1,1)$ given by C_1: $r(t) = ti + t^2j$, $0 \le t \le 1$; C_2: $r(t) = i + j + tk$, $0 \le t \le 1$.

$$\int_C (x+\sqrt{y}-z^4)\,ds = \int_{C_1} (x+\sqrt{y}-z^4)\,ds + \int_{C_2} (x+\sqrt{y}-z^4)\,ds$$

C_1

$r(t) = ti + t^2j$ $0 \le t \le 1$
$r'(t) = 1i + 2tj$
$ds = |v'(t)|$
$= \sqrt{1+4t^2}\,dt$

$\int_0^1 (t+t-0^4)\sqrt{1+4t^2}\,dt$

$\int_0^1 2t\sqrt{1+4t^2}\,dt$ $u = 1+4t^2$, $du = 8t\,dt$, $t\,dt = \frac{1}{8}du$

$2\int_0^1 \frac{\sqrt{u}}{8}\,du = 2\left[\frac{u^{\frac{1}{2}+1}}{8\cdot\frac{1}{2}+1}\right]_0^1 \quad 2\left[\frac{u^{\frac{3}{2}}}{12}\right]_0^1 = \left[\frac{1}{6}(1+4t^2)^{\frac{3}{2}}\right]_0^1$

$= \frac{1}{6}(5)^{\frac{3}{2}} - 1 = \frac{1}{6}(5\sqrt{5}-1)$

C_2

$r(t) = i + j + tk$ $0 \le t \le 1$
$v'(t) = 1$
$ds = |v'(t)|$
$= \sqrt{1}\,dt = 1\,dt$

$\int_0^1 (1+1-t^4)\,dt = \int_0^1 (2-t^4)\,dt$
$= \left[2t - \frac{t^5}{5}\right]_0^1 = \frac{9}{5}$

$\int_C (x+\sqrt{y}-z^4)\,ds = \frac{1}{6}(5\sqrt{5}-1) + \frac{9}{5} = \frac{5\sqrt{5}}{6} - \frac{1}{6} + \frac{9}{5}$

$= \frac{5\sqrt{5}}{6} - \frac{49}{30}$

16.2 Vector Fields and Line Integrals: Work, Circulation, and Flux

Find the line integrals of $F = yi + 4xj + 2zk$ from $(0,0,0)$ to $(1,1,1)$ over the straight-line path C_1: $r(t) = ti + tj + tk$, $0 \le t \le 1$.

Vector function $F = yi + 4xj + 2zk$

C_1: $r(t) = ti + tj + tk$ $0 \le t \le 1$

since $x=t$, $y=t$, $z=t$ $F(r(t)) = ti + 4tj + 2tk$

FLOW

$$\int_C F\cdot dr = \int_a^b \left(F(r(t))\cdot\frac{dr}{dt}\right)dt \qquad \frac{dr}{dt} = \frac{d}{dt}(r(t)) = i+j+k$$

$\int_0^1 (ti + 4tj + 2tk)\cdot(i+j+k)\,dt$

$\int_0^1 (t+4t+2t)\,dt = \int_0^1 7t\,dt = \left[7\frac{t^2}{2}\right]_0^1 = \frac{7}{2}$

Find the work done by F over the curve in the direction of increasing t. $F = 3xyi + 2yj - 3yzk$

$r(t) = ti + t^2j + tk$ $0 \le t \le 1$
$r'(t) = i + 2tj + k$ $x=t$ $y=t^2$ $z=t$

WORK

$F = 3\cdot t\cdot t^2 i + 2t^2 j - 3\cdot t^2\cdot t\,k$
$= 3t^3 i + 2t^2 j - 3t^3 k$

$\text{work} = \int_a^b F\cdot\frac{dr}{dt}\,dt = \int_0^1 (3t^3i + 2t^2j - 3t^3k)\cdot(i+2tj+k)\,dt$

$= \int_0^1 (3t^3+4t^3-3t^3)\,dt = \int_0^1 4t^3\,dt = [t^4]_0^1 = 1$

Evaluate $\int_C xy\,dx + (x+y)\,dy$ along the curve $y = 2x^2$ from $(-3,18)$ to $(-2,8)$.

$$\int_a^b (M\,dx + N\,dy) = \int_a^b \left(\overset{xy}{M}\frac{dx}{dt} + \overset{x+y}{N}\frac{dy}{dt}\right)dt$$

Parameterize
let $x = t$ $y = 2t^2$
$\frac{dx}{dt} = 1$ $\frac{dy}{dt} = 4t$
x limits
$-3 \le t \le -2$

$M = xy$ $N = x+y$
$M = 2t^3$ $N = t + 2t^2$

FLUX

$\int_C xy\,dx + (x+y)\,dy$

$\int_{-3}^{-2} \left(xy\frac{dx}{dt} + (x+y)\frac{dy}{dt}\right)dt$

$\int_{-3}^{-2} (2t^3\cdot 1 + (t+2t^2)4t)\,dt$

$\int_{-3}^{-2} (2t^3 + 4t^2 + 8t^3)\,dt$

$\int_{-3}^{-2} (10t^3 + 4t^2)\,dt$

$= \left[10\frac{t^4}{4} + 4\frac{t^3}{3}\right]_{-3}^{-2}$

$= \left[10\frac{(-2)^4}{4} + 4\frac{(-2)^3}{3}\right] - \left[10\frac{(-3)^4}{4} + 4\frac{(-3)^3}{3}\right]$

$= \left(\frac{88}{3} - \frac{333}{2}\right)$

$= -\frac{823}{6}$

Find the flux of the field $F = -2xi - 2yj$ around and across the closed semicircular path that consists of the semicircular arch $r(t) = (a\cos t)i + (a\sin t)j$, $0 \le t \le \pi$

FLUX

$M = -2x$ $N = -2y$
$= -2a\cos t$ $= -2a\sin t$
$dx = (-a\sin t)\,dt$ $dy = (a\cos t)\,dt$ $0 \le t \le \pi$

$$\text{Flux} = \oint M\,dy - N\,dx$$

$\int M\,dy - N\,dx = \int_0^\pi (-2a\cos t\,(a\cos t) + 2a\sin t\,(-a\sin t))\,dt$

$\int_0^\pi (-2a^2\cos^2 t - 2a^2\sin^2 t)\,dt = \int_0^\pi (-2a^2)(\cos^2 t + \sin^2 t)\,dt$

$\int_0^\pi -2a^2\cdot 1\,dt = [-2a^2t]_0^\pi = -2a^2\pi$

16.3 Path Independence, Conservative Fields, and Potential Functions.

Determine if the field $F = 16yzi + 16xzj + 16xyk$ is conservative or not conservative.

$M = 16yz$ $N = 16xz$ $P = 16xy$

F is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x},\ \text{and}\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$\frac{\partial P}{\partial y} = 16x$ ✓ $\frac{\partial M}{\partial z} = 16y$ ✓ $\frac{\partial N}{\partial x} = 16z$ ✓
$\frac{\partial N}{\partial z} = 16x$ $\frac{\partial P}{\partial x} = 16y$ $\frac{\partial M}{\partial y} = 16z$

$F = 16yzi + 16xzj + 16xyk$ is conservative

CONSERVATIVE

Show that the value of the integral below does not depend on the path taken from A to B.

$\int_A^B z^2\,dx + 2y\,dy + 2xz\,dz$

$M = z^2$ $N = 2y$ $P = 2xz$
$\frac{\partial P}{\partial y} = 0$ $\frac{\partial M}{\partial z} = 2z$ $\frac{\partial N}{\partial x} = 0$
$\frac{\partial N}{\partial z} = 0$ $\frac{\partial P}{\partial x} = 2z$ $\frac{\partial M}{\partial y} = 0$

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$ ✓

INDEPENDENT

therefore, the function $F = z^2dx + 2ydy + 2xzdz$ is conservative. so, its integration is path independent.

$\int_A^B z^2dx + 2ydy + 2xzdz = \left[0 + y^2 + 2x\cdot\frac{z^2}{2}\right]_A^B = [xz^2 + y^2]_A^B$

Find the potential function f for the field $F = 4xi + 4yj + 9zk$

$\frac{\partial f}{\partial x} = 4x$ $\frac{\partial f}{\partial y} = 4y$ $\frac{\partial f}{\partial z} = 9z$

Integrate $\frac{\partial f}{\partial x} \rightarrow \int 4x\,dx = 2x^2 + g(y,z) \rightarrow f = 2x^2 + g(y,z)$

Partially differentiate $\frac{\partial f}{\partial y} \rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} \rightarrow \frac{\partial g}{\partial y} = 4y$

Integrate $\frac{\partial g}{\partial y} \rightarrow \int 4y\,dy = 2y^2 + h(z) \rightarrow g = 2y^2 + h(z)$

Partially differentiate $\frac{\partial f}{\partial z} \rightarrow \frac{\partial f}{\partial z} = \frac{\partial h}{\partial z} \rightarrow \frac{\partial h}{\partial z} = 9z$

Integrate $\frac{\partial g}{\partial y} \rightarrow \int 9z\,dz = \frac{9}{2}z^2 + C \rightarrow h = \frac{9}{2}z^2 + C$

Join f, g and h $f = 2x^2 + 2y^2 + \frac{9}{2}z^2 + C$

Find a potential function for $F = \frac{2x}{y}i + \frac{10-x^2}{y^2}j$ $\{(x,y): y>0\}$

$\frac{\partial f}{\partial x} = \frac{2x}{y}$ $\frac{\partial f}{\partial y} = \frac{10-x^2}{y^2}$

POTENTIAL

$f(x,y) = \int \frac{2x}{y}\,dx = \frac{x^2}{y} + g(y)$

$\frac{\partial f}{\partial y} = \cancel{\frac{-x^2}{y^2}} + g'(y) = \frac{10-\cancel{x^2}}{y^2}$ $x^2\cdot y^{-1} = -x^2y^{-2} = -\frac{x^2}{y^2}$

$g'(y) = \frac{10}{y^2}$

$g(y) = \int \frac{10}{y^2}\,dy = \frac{-10}{y}$ $\frac{10}{y^2} = 10\cdot\frac{y^{-2+1}}{-2+1} = -10y^{-1} = \frac{-10}{y}$

$f(x,y) = \frac{x^2}{y} - \frac{10}{y} + C$

Show that the differential form in the integral below is exact. Then evaluate the integral.

use: $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$\int_{(0,0,0)}^{(4,-2,-5)} 12x\,dx + 16y\,dy + 10z\,dz$ $M = 12x$ $N = 16y$ $P = 10z$

$\frac{\partial P}{\partial y} = 0$ ✓ $\frac{\partial M}{\partial z} = 0$ ✓ $\frac{\partial N}{\partial x} = 0$ ✓
$\frac{\partial N}{\partial z} = 0$ $\frac{\partial P}{\partial x} = 0$ $\frac{\partial M}{\partial y} = 0$

the differential form $12x\,dx + 16y\,dy + 10z\,dz$ is exact.

EXACT

$f(x,y,z) = \int 12x\,dx = 6x^2 + g(y,z)$ $f(x,y,z) = \int_{(0,0,0)}^{(4,-2,-5)} 12x\,dx + 16y\,dy + 10z\,dz$

$g(y,z) = \int 16y\,dy = 8y^2 + h(z)$ $= [6x^2 + 8y^2 + 5z^2 + C]_{(0,0,0)}^{(4,-2,-5)}$

$h(z) = \int 10z\,dz = 5z^2 + C$ $= 253$

16.4 Green's Theorem in The Plane

Find the k-component of (curl F) for the following vector field on the plane.

$F = (x+4y)i + (9xy)j$ CURL F

$(\text{curl}\,F)\cdot k = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial}{\partial x}9xy - \frac{\partial}{\partial y}x + 4y$

$= 9y - 4$

Use Green's Theorem to find the counterclockwise circulation and outward flux for the field $F = (y^2-5x^2)i + (5x^2+y^2)j$ and curve C: the square bounded by $y=0$, $x=3$, and $y=x$

$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = (y^2-5x^2) + (5x^2+y^2) = -10x + 2y$ $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (5x^2+y^2) - (y^2-5x^2) = 10x - 2y$

Flux divergence (Normal form)

$$\oint M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy$$

$\int_0^x\int_0^3 (-10x+2y)\,dx\,dy$

Integrate with respect to y first.

$\int_0^3\int_0^x (-10x+2y)\,dy\,dx$

$\int_0^3 [-10xy + y^2]_0^x\,dx$

$\int_0^3 (-10x^2 + x^2)\,dx$

$\int_0^3 -9x^2\,dx = [-3x^3]_0^3 = -81$

Circulation curl (tangential form)

$$\oint M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

$\int_0^x\int_0^3 (10x-2y)\,dx\,dy$

Integrate with respect to y first.

$\int_0^3\int_0^x (10x-2y)\,dy\,dx$

$\int_0^3 [10xy - y^2]_0^x\,dx$

$\int_0^3 (10x^2 - x^2)\,dx$

$\int_0^3 9x^2\,dx = [3x^3]_0^3 = 81$

Circulation can be zero.

16.5 Surfaces and Area.

Find a parameterization of the paraboloid $225z = 9x^2 + 25y^2$, $z \le 4$.

Parameterization of a curve $r(u,v) = f(u,v)i + g(u,v)j + h(u,v)k$

$z = \frac{9x^2}{225} + \frac{25y^2}{225} = \frac{x^2}{\frac{225}{9}} + \frac{y^2}{\frac{225}{25}} \rightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1$ since $z \le 4$.

$u = \theta$ $v = r$ $x = r\cos\theta$ $y = r\sin\theta$

$\frac{x^2}{25} = 1$ $\frac{y^2}{9} = 1$
$x^2 = 25$ $y^2 = 9$
$x = 5$ $y = 3$
$x = 5\cos\theta$ $y = 3\sin\theta$
since $u = \theta$ and $v = r$
$x = 5v\cos u$, $0 \le \theta \le 2\pi$
$y = 3v\sin u$

Find z $z = \frac{x^2}{25} + \frac{y^2}{9}$

$z = \frac{(5v\cos u)^2}{25} + \frac{(3v\sin u)^2}{9}$

$z = \frac{\cancel{25}v^2\cos^2u}{\cancel{25}} + \frac{\cancel{9}v^2\sin^2u}{\cancel{9}}$

$z = v^2\cos^2u + v^2\sin^2u$

$z = v^2(\cos^2u + \sin^2u) = v^2$

u-limits
since $u = \theta$ and $0 \le \theta \le 2\pi$
then, $0 \le u \le 2\pi$

v-limits
since $0 \le z \le 4$ and $z = v^2$
solving for zero when $z=0$ $v=0$
and $z=4$ $v=2$ thus, $0 \le v \le 2$.

therefore,

$r(u,v) = (5v\cos u)i + (3v\sin u)j + v^2k$ $0 \le u \le 2\pi$, $0 \le v \le 2$.

Use a parameterization to express the area of the surface as a double integral. The portion of the cone $z = 4\sqrt{x^2+y^2}$ between the planes $z=0$ and $z=16$.

Let $u=r$ and $v=\theta$ and use cylindrical coordinates to parametize the surface.

$x = r\cos\theta$ $y = r\sin\theta$ $0 \le \theta \le 2\pi$ $v = \theta$ $u = r$

thus, $x = u\cos v$ $y = u\sin v$

Find z

$z = 4\sqrt{(u\cos v)^2 + (u\sin v)^2}$
$z = 4\sqrt{(u^2\cos^2v) + (u^2\sin^2v)}$
$z = 4\sqrt{u^2(\cos^2v + \sin^2v)}$
$z = 4\sqrt{u^2} = 4u$

u-limits $u = r$
since $0 \le z \le 16$ and $z = 4u$
Solving for zero when $z=0$, $u=0$ and $z=16$, $u=4$.
thus, $0 \le u \le 4$

v-limits $v = \theta$
$0 \le v \le 2\pi$

thus, $r(u,v) = (u\cos v)i + (u\sin v)j + (4u)k$ $0 \le u \le 4$, $0 \le v \le 2\pi$

$r_u = (\cos v)i + (\sin v)j + (4)k$ $r_v = (-u\sin v)i + (u\cos v)j + 0k$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 4 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \begin{vmatrix} \sin v & 4 \\ u\cos v & 0 \end{vmatrix} i - \begin{vmatrix} \cos v & 4 \\ -u\sin v & 0 \end{vmatrix} j + \begin{vmatrix} \cos v & \sin v \\ -u\sin v & u\cos v \end{vmatrix} k$$

$= -4u\cos v - 4u\sin v + (u\cos^2v - u\sin^2v)$
$= -4u\cos v - 4u\sin v + u$

$|r_u \times r_v| = \sqrt{(-4u\cos v)^2 + (-4u\sin v)^2 + u^2} = \sqrt{(16u^2\cos^2v) + (16u^2\sin^2v) + u^2}$
$= \sqrt{16u^2(\cos^2v + \sin^2v) + u^2} = \sqrt{16u^2 + u^2} = \sqrt{17u^2} = \sqrt{17}\,u$

$$A = \int_c^d\int_a^b |r_u \times r_v|\,du\,dv$$

AREA

$= \int_0^{2\pi}\int_0^4 \sqrt{17}\,u\,du\,dv = \sqrt{17}\int_0^{2\pi}\left[\frac{u^2}{2}\right]_0^4 dv = \sqrt{17}\int_0^{2\pi} 8\,dv = \sqrt{17}\,[8v]_0^{2\pi} = 16\pi\sqrt{17}$

Find the area of the region cut from the plane $4x + y + 8z = 7$ by the cylinder whose walls are $x = y^2$ and $x = 2 - y^2$.

$f(x,y,z) = 4x + y + 8z - 7$ $|\nabla f| = \sqrt{4^2 + 1^2 + 8^2}$ $|\nabla f\cdot p| = |(4i + 1j + 8k)\cdot(k)|$

$$\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

$= \sqrt{81} = 9$ $= 8$

$\nabla f = 4i + 1j + 8k$ $p = k$

$$\iint_R \frac{|\nabla f|}{|\nabla f\cdot p|}\,dA = \iint_R \frac{9}{8}\,dA = \frac{9}{8}\int_{-1}^1 2 - 2y^2\,dy$$

y-limits (y-intersect)
$x = y^2$ $x = 2 - y^2$
$y^2 = 2 - y^2$
$2y^2 = 2$
$y^2 = 1$
$y = \pm 1$

$\frac{9}{8}\int_{-1}^1\int_{y^2}^{2-y^2} dx\,dy = \frac{9}{8}\left[2y - \frac{2}{3}y^3\right]_{-1}^1$

$\frac{9}{8}\int_{-1}^1 [x]_{y^2}^{2-y^2}\,dy = \frac{9}{8}\left[2\cdot 1 - \frac{2}{3}1^3 - 2\cdot{-1} - \frac{2}{3}{-1}^3\right]$

$\frac{9}{8}\int_{-1}^1 [2 - y^2 - y^2]\,dy = \frac{9}{8}\left[\frac{4}{3} + \frac{4}{3}\right] = 3$

AREA

16.6 Surface Integrals.

Integrate the given function over the given surface.
$G(x,y,z)=z$ over the parabolic cylinder $y=z^2$, $0 \le x \le 2$, $0 \le z \le \frac{\sqrt{3}}{2}$

$v = xi + yj + zk \qquad v_x = 1i$

$= xi + z^2 j + zk \qquad v_z = 2zj + 1k$

$$v_x \times v_z = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 2z & 1 \end{vmatrix} = (0-0)i - (1-0)j + (2z-0)k = -1j + 2zk$$

$$|v_x \times v_z| = \sqrt{(-1)^2 + (2z)^2} = \sqrt{1+4z^2}$$

$$\iint_S G(x,y,z)\,d\sigma = \int_0^2 \int_0^{\frac{\sqrt{3}}{2}} z \cdot \sqrt{1+4z^2}\,dz\,dx$$

$u = 1 + 4z^2$, $du = 8z\,dz$, $z\,dz = \frac{1}{8}du$, $v \to 0 \quad u \to 1$, $v \to \frac{\sqrt{3}}{2} \quad u \to 4$

$$\frac{1}{8}\int_0^2\int_1^4 \sqrt{u}\,du\,dx = \frac{1}{8}\int_0^2 \left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^4 dx = \frac{1}{8}\int_0^2 \left[\frac{2}{3}4^{\frac{3}{2}} - \frac{2}{3}1^{\frac{3}{2}}\right]$$

$$= \frac{1}{8}\int_0^2 \frac{14}{3}\,dx = \frac{1}{8}\frac{14}{3}\left[x\right]_0^2 = \frac{7}{6}$$

use parameterization to find the flux $\iint_S F \cdot n\,d\sigma$ of $F = z^2 i + xj - 2zk$ in the outward direction (normal away from the x-axis) accross the surface cut from the parabolic cylinder $z = 1 - y^2$ by the planes $x=0$, $x=1$ and $z=0$.

Let the parameterization be
$v(x,y) = xi + yj + (1-y^2)k$
$v_x = 1i \qquad v_y = 1j - 2yk$

plug $z=0$ into $z = 1-y^2$
$0 = 1 - y^2$
$y = \pm 1$ and $0 \le x \le 1$

① $v_x \times v_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = (0-0)i - (-2y-0)j + (1-0)k = 2yj + 1k$

② $|v_x \times v_y| = \sqrt{(2y)^2 + 1^2} = \sqrt{4y^2+1}$

③ $F \cdot n\,d\sigma = \left[z^2 i + xj - 2(1-y^2)k\right] \cdot \left[2yj + 1k\right] dy\,dx = \left[2xyj - 2(1-y^2)k\right] dy\,dx$

$$\iint_S F \cdot n\,d\sigma = \iint_R F \cdot (r_x \times r_y)\,dx\,dy$$

$$\int_0^1\int_{-1}^1 (2xyj + 2y^2 - 2)\,dy\,dx = \int_0^1 \left[2x\frac{y^2}{2} + \frac{2}{3}y^3 - 2y\right]_{-1}^1 dx$$

$$\int_0^1 (x + \tfrac{2}{3} - 2) - (x - \tfrac{2}{3} + 2)\,dx$$

$$\int_0^1 (\cancel{x} + \tfrac{2}{3} - 2 - \cancel{x} + \tfrac{2}{3} - 2)\,dx = \int_0^1 (\tfrac{4}{3} - 4)\,dx = \left[-\tfrac{8}{3}x\right]_0^1 = -\frac{8}{3}$$

Find the surface integral of the field $F(x,y,z) = -i + 4j + 4k$ across the rectangular surface $z = 0$, $0 \le x \le 3$, $0 \le y \le 2$ in the k direction.

$P = k$
$g(x,y,z) = z$

$$\text{Flux} = \iint_S F \cdot n\,d\sigma, \quad n = \pm\frac{\nabla g}{|\nabla g|}, \quad d\sigma = \frac{\nabla g}{|\nabla g \cdot p|}$$

$\nabla g = 0i + 0j + 1k \qquad |\nabla g| = \sqrt{1^2} = 1$

$|\nabla g \cdot p| = |(0i + 0j + 1k)\cdot(k)| = 1$

$n = \frac{\nabla g}{|\nabla g|} = \pm\frac{1k}{1} = \pm k$ choose + direction

$F \cdot n = (-i + 4j + 4k)\cdot(k) = 4$

$d\sigma = \frac{\nabla g}{|\nabla g \cdot p|} = \frac{1}{1} = 1$

$$= \int_0^3\int_0^2 4\,dy\,dx \qquad \int_0^3 \left[4y\right]_0^2 dx \qquad \int_0^3 8\,dx = \left[8x\right]_0^3 = 24$$

16.7 Stokes' Theorem.

Find the curl of the vector field $F = (x^7yz)i + (xy^2z)j + (xyz^8)k$
(M, N, P)

$$\text{curl } F = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)i + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)j + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)k$$

$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(xyz^8) = xz^8 \qquad \frac{\partial N}{\partial z} = \frac{\partial}{\partial z}(xy^2z) = xy^2 \qquad = (xz^8 - xy^2)i$

$\frac{\partial M}{\partial z} = \frac{\partial}{\partial z}(x^7yz) = x^7y \qquad \frac{\partial P}{\partial x} = \frac{\partial}{\partial x}(xyz^8) = yz^8 \qquad + (x^7y - yz^8)j$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(xy^2z) = y^2z \qquad \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^7yz) = x^7z \qquad + (y^2z - x^7z)k$

$$= x(z^8 - y^2)i + y(x^7 - z^8)j + z(y^2 - x^7)k$$

Use the surface integral in Stokes' Theorem to calculate the circulation of the field $F = x^2 i + 5xj + z^2 k$ around the curve C: the ellipse $25x^2 + 9y^2 = 10$ in the xy-plane, counterclockwise when viewed from above.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 5x & z^2 \end{vmatrix} = \left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(5x)\right]i - \left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right]j + \left[\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(x^2)\right]k = (0-0)i - (0-0)j + (5-0)k = 5k$$

$$\oint_C F \cdot dr = \iint_S \nabla \times F \cdot n\,d\sigma \qquad n = k,\ d\sigma = dx\,dy$$

$$= \iint_S 5k \cdot k\,dx\,dy = \iint_R 5\,dA = 5\pi ab$$

$$= 5\pi\left(\frac{\sqrt{10}}{5}\right)\left(\frac{\sqrt{10}}{3}\right) = 5\pi\left(\frac{10}{15}\right) = \frac{10\pi}{3}$$

$y = 0$: $25x^2 = 10$, $x^2 = \frac{10}{25}$, $x^2 = \sqrt{\frac{10}{25}}$, a $x = \frac{\sqrt{10}}{5}$

$x = 0$: $9y^2 = 10$, $y^2 = \frac{10}{9}$, $y^2 = \sqrt{\frac{10}{9}}$, b $y = \frac{\sqrt{10}}{3}$

16.8 The Divergence Theorem.

Find the divergence of the field $F = (-6x + y + 4z)i + (3x - y + 7z)j + (3x - y + 7z)k$
(M, N, P)

$$\text{div } F = \frac{\partial}{\partial x}(-6x + y + 4z) + \frac{\partial}{\partial y}(3x - y + 7z) + \frac{\partial}{\partial z}(3x - y + 7z)$$

$$\text{div } F = \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = -6 - 1 + 7 = 0$$

Find the divergence of the spin field $F = \frac{-4yi + 4xj}{(x^2+y^2)^{1/2}}$

$M = -\frac{4y}{(x^2+y^2)^{1/2}} \qquad N = \frac{4x}{(x^2+y^2)^{1/2}}$

$$\text{div } F = \frac{\partial}{\partial x} - 4y(x^2+y^2)^{-1/2} + \frac{\partial}{\partial y} 4x(x^2+y^2)^{-1/2}$$

$$= -4y\left[-\frac{1}{\cancel{2}}(x^2+y^2)^{-\frac{3}{2}} \cdot (\cancel{2}x)\right] + 4x\left[-\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}} \cdot (2y)\right]$$

$$= \frac{4xy}{(x^2+y^2)^{3/2}} - \frac{4xy}{(x^2+y^2)^{3/2}} = \frac{4xy - 4xy}{(x^2+y^2)^{3/2}} = 0$$

use the divergence theorem to find the outward flux of F across the boundary of the region D. $F = (3y - x)i + (2z - y)j + (4y - 4x)k$
(M, N, P)
D: the cube bounded by the planes $x = \pm 3$, $y = \pm 3$, and $z = \pm 3$.

$$\text{div } F = \frac{\partial}{\partial x}(3y - x) + \frac{\partial}{\partial y}(2z - y) + \frac{\partial}{\partial z}(4y - 4x)$$

$$\text{div } F = \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = -1 - 1 + 0 = -2$$

$$\iint_S F \cdot n\,d\sigma = \iiint_D \nabla \cdot F\,dV = \iiint_D -2\,dV = \int_{-3}^3\int_{-3}^3\int_{-3}^3 -2\,dx\,dy\,dz$$

$$= -2\int_{-3}^3\int_{-3}^3 \left[x\right]_{-3}^3 dy\,dz = -2\int_{-3}^3\int_{-3}^3 6\,dy\,dz = -2\int_{-3}^3 \left[6y\right]_{-3}^3 dz = -2\int_{-3}^3 36\,dz = -2\left[36z\right]_{-3}^3 = -432$$

use the divergence theorem to find the outward flux of $F = 7yi + 5xyj - 4zk$ (M, N, P) across the boundary of the region D: the region inside the solid cylinder $x^2 + y^2 \le 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$.

$$\text{div } F = \frac{\partial}{\partial x}7y + \frac{\partial}{\partial y}5xy + \frac{\partial}{\partial z}-4z = 0 + 5x - 4 = 5x - 4$$

$x = r\cos t \qquad y = r\sin t \qquad r^2 = x^2 + y^2 = 4 \rightarrow r = 2 \qquad z = r^2$

D: $0 \le r \le 2$, $\quad 0 \le t \le 2\pi$ and $0 \le z \le r^2$

$$\iint_S F \cdot n\,d\sigma = \iiint_D \nabla \cdot F\,dV$$

$$\iiint_D 5x - 4\,dV$$

$$\int_0^{2\pi}\int_0^2\int_0^{r^2} (5r\cos t - 4)r\,dz\,dr\,dt$$

$$\int_0^{2\pi}\int_0^2\int_0^{r^2} 5r^2\cos t - 4r\,dz\,dr\,dt$$

$$\int_0^{2\pi}\int_0^2 (5r^2\cos t - 4r)\left[z\right]_0^{r^2} dr\,dt$$

$$\int_0^{2\pi}\int_0^2 (5r^4\cos t - 4r^3)\,dr\,dt$$

$$= \int_0^{2\pi} \left[r^5\cos t - r^4\right]_0^2 dt$$

$$\int_0^{2\pi} 32\cos t - 16\,dt$$

$$\left[32\sin t - 16t\right]_0^{2\pi}$$

$32\sin 2\pi - 16\,2\pi - 32\sin 0 - 16\,0$

$0 - 32\pi - 0 \qquad \sin 2\pi = \sin 0 = 0$

-32π

$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$ Direction $\vec{v} = \frac{\vec{v}}{|\vec{v}|}$

Component form $K\langle v_1, v_2 \rangle = \langle Kv_1, Kv_2 \rangle$

Magnitude length $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ See Area for cross product.

angle between vectors $\theta = \cos^{-1}\frac{u \cdot v}{|u| \cdot |v|}$ RADIANS $\quad \text{proj}_v u = \left(\frac{u \cdot v}{|v|^2}\right) v$

EXAM 1

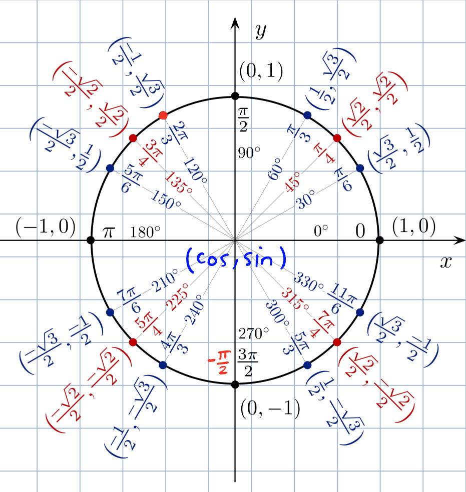
Line passing through a point and it's parallel to a vector $i + j + k$
Parameterize $v_1 i + v_2 j + v_3 k \qquad (x_0, y_0, z_0)$
$x = x_0 + tv_1 \qquad y = y_0 + tv_2 \qquad z = z_0 + tv_3$

Line passing through a point and it's perpendicular to the plane $Ax + By + Cz = D$ / Convert $Ax + By + Cz = D$ to normal vector $i + j + k$ / Parameterize

Find parametric equation for the tangent line
1. find $v(t)$ at $t=0$ to get tan line (x,y,z) $x = 3 + 0t = 3$
2. find $v(t) = \frac{dr}{dt} = \frac{dr}{dt}t = \frac{dr}{dt}$ of $v(t)$ $\langle 3, 3t, 4t \rangle$
3. evaluate $v(t)$ at $t=0$.
4. parametize tan line (x,y,z) parallel to $v(0)$

solve the initial value problem for $\vec{r}$ as a vector function of t
1. integrate $\vec{r}$. 2. evaluate $\vec{v}(0) = \vec{r}(t)$ to find C.
3. substitute C in $\vec{r}(t)$.

Differentiation Rules	
Constant Rule	$\frac{d}{dx}[c] = 0$
Power Rule	$\frac{d}{dx}x^n = nx^{n-1}$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$
Chain Rule	$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$



Reference

COMMON FACTORING EXAMPLES

$x^2 - a^2 = (x+a)(x-a)$

$x^2 + 2ax + a^2 = (x+a)^2$

$x^2 - 2ax + a^2 = (x-a)^2$

$x^2 + (a+b)x + ab = (x+a)(x+b)$

$x^3 + 3ax^2 + 3a^2x + a^3 = (x+a)^3$

$x^3 + a^3 = (x+a)(x^2 - ax + a^2)$

$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$

$x^{2n} - a^{2n} = (x^n - a^n)(x^n + a^n)$

$u = 3y$, $du = 3\,dy$, $dy = \frac{1}{3}du$

Adjust integral boundaries
$y = 1 \rightarrow u = 3$
$y = \ln 5 \rightarrow u = 3\ln 5$

$\int \frac{1}{u}du = \ln(u) \qquad \int e^y dy = e^y$

$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} \qquad a^{\log_a(b)} = b$

$e^1 = e \qquad e^{x+y} = e^x \cdot e^y$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Derivative	Integral (Antiderivative)
$\frac{d}{dx}n = 0$	$\int 0\,dx = C$
$\frac{d}{dx}x = 1$	$\int 1\,dx = x + C$
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
$\frac{d}{dx}e^x = e^x$ $\quad \frac{d}{dx}e^u = e^u \cdot u'$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}\ln x = \frac{1}{x}$ $\quad \frac{d}{dx}\ln u = \frac{u'}{u}$	$\int \frac{1}{x}\,dx = \ln x + C$
$\frac{d}{dx}n^x = n^x \ln x$	$\int n^x\,dx = \frac{n^x}{\ln n} + C$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \tan x \sec x\,dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\int \cot x \csc x\,dx = -\csc x + C$
$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$
$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$	$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + C$
$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \arctan x + C$
$\frac{d}{dx}\text{arccot}\, x = -\frac{1}{1+x^2}$	$\int -\frac{1}{1+x^2}\,dx = \text{arccot}\, x + C$
$\frac{d}{dx}\text{arcsec}\, x = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \text{arcsec}\, x + C$
$\frac{d}{dx}\text{arccsc}\, x = -\frac{1}{x\sqrt{x^2-1}}$	$\int -\frac{1}{x\sqrt{x^2-1}}\,dx = \text{arccsc}\, x + C$

Degree	Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\text{cosec}\,\theta$	$\sec\theta$	$\cot\theta$
0°	0	0	1	0	-	1	-
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$ $\frac{1}{\sqrt{3}}$	2	$\frac{2\sqrt{3}}{2}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{2}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	-	1	-	0
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2	$-\frac{\sqrt{3}}{3}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	2	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$
180°	π	0	-1	0	-	-1	-
270°	$\frac{3\pi}{2}$	-1	0	-	-1	-	0
360°	2π	0	1	0	-	1	-

FUNDAMENTAL IDENTITIES

$\csc\theta = \frac{1}{\sin\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta \qquad \tan(-\theta) = -\tan\theta$

$\sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \sin^2\theta + \cos^2\theta = 1 \qquad \sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$

EXAM 2

Find the curve's unit tangent vector $\vec{r}(t)$

unit tangent vector $T = \frac{dr}{ds} = \frac{dr/dt}{ds/dt} = \frac{v}{|v|}$

Length $L = \int_a^b |\vec{v}|\,dt$

$|\vec{v}| = \sqrt{V_1^2 + V_2^2 + V_3^2}$

1. Find v $i + j + k$ (derivative with respect to t)
2. Find $|\vec{v}|$ Square each term
3. Simplify the radical $\cos^2\theta + \sin^2\theta = 1$
4. Find T $i + j + k$
5. Find length L $3\sqrt{2}\,t \Big|_0^{\pi}$ $3\sqrt{2}\,\pi - 3\sqrt{2}(0)$

product rule $(f \cdot g)' = f' \cdot g + f \cdot g'$ $3t \cos t$

Find the arc length parameter along the given curve from P where $t = 0$.

1. Find $v(t)$ (derivative) $i + j + k$
2. Find length of velocity vector $|\vec{v}(t)|$
3. Find arc length $s(t)$ $\int_0^t |\vec{v}(t)|\,dt$

0 to t then 0 to π then diff. between $t = 0$ and $t = \pi$

Calculate the length of one turn of the helix with the following parameterizations

1. Find $\vec{v}$, derivative $r(t)$. $i + j + k$
2. Find length of velocity vector $|\vec{v}(t)|$
3. Find arc length parameter $L = \int_{t_0}^{t} |\vec{v}(t)|\,dt$

may need chain rule $f'(g(x)) = f'(g(x))\,g'(x)$

Find T, N, and κ for the plane curve $\vec{r}(t)$

1. Find velocity vector $\vec{v} = r'(t)$ $i + j + k$
2. Find length $|\vec{v}|$ $1 + \tan^2 t = \sec^2 t$ $\cos t = \frac{1}{\sec t}$
3. Find unit tangent vector $T = \frac{v}{|v|}$ may need chain rule
4. Find dT/dt (derivative of $T(t)$)
5. Find $|dT/dt|$
6. Principal normal vector $N = \frac{dT/dt}{|dT/dt|}$ $i\,j\,k$
7. Find curvature $\kappa = \frac{1}{|v|} \cdot \left|\frac{dT}{dt}\right|$

Find the total curvature of the helix

1. steps 1 to 5 $K = \int_{t_0}^{t_1} \left|\frac{dT}{dt}\right| dt$
2. Perform integration

Find the total curvature of the parabola

1. Parametize $y = 2x^2 \to r(t) = i + 2t^2 j$
2. steps 1 to 5 $\int \frac{1}{1+x^2} = \tan^{-1} x$

Find an equation for the circle of curvature

1. find $\vec{v}$ and $|\vec{v}|$ $k = \frac{|x'y'' - y'x''|}{\left[1 + (y'(x))^2\right]^{3/2}}$

maximum curvature $k(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}$

Function	Domain	Range
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$z = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	Entire plane	$[-1, 1]$

$\geq$ [

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Function	Domain	Range
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$w = \frac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$

Find and sketch the level curves $f(x,y) = c$.

1. substitute c in $f(x,y) = c$.
2. Solve for y

Find lim 1. if f(x) is defined, plug in values.

else: factor and simplify, then plug values.

two path test 1. find lim $x \to 0^+$ at $y = x$ then $x \to 0^-$ at $y = x$.

for $x \to 0^+$ $|x|$ for $x \to 0^-$ $-x$.

1. if $f(x,y) = \frac{2x^2y}{x^4 + y^2}$ undefined at (0,0) use $y = kx^2$
2. simplify then find value at $k = 1, k = 0$ if lims ≠ then lim does not exist.

Find $\partial f/\partial x$ and $\partial f/\partial y$.

for $\partial f/\partial x$ treat y as a constant $4y^2 = 0$, $8xy = 8y$

for $\partial f/\partial y$ treat x as a constant $5x^9 = 0$, $8xy = 8x$

may need chain rule $f'(g(x)) = f'(g(x))\,g'(x)$

product rule $(f \cdot g)' = f' \cdot g + f \cdot g'$

$\frac{\partial}{\partial x}(a^x) = a^x \ln a$

second order partial derivatives

$\frac{\partial^2 g}{\partial x^2}$ find $\frac{\partial g}{\partial x}'$ by taking derivative of $g(x,y)$ treating y as a constant. then $\frac{\partial g}{\partial x}''$

$\frac{\partial^2 g}{\partial y \partial x} \to \frac{\partial}{\partial y}\frac{\partial g}{\partial x}$ take derivative of dg/dx treating x as a constant.

$\frac{\partial^2 g}{\partial y^2}$ find $\frac{\partial g}{\partial y}'$ by taking derivative of $g(x,y)$ then treating x as a constant. $\frac{\partial g}{\partial y}''$

$\frac{\partial^2 g}{\partial x \partial y} \to \frac{\partial}{\partial x}\frac{\partial g}{\partial y}$ take derivative of dg/dy treating y as a constant.

use the limit definition of partial derivatives to compute the partial derivative of $f(x,y) = 7 - 2x + 5y - 4x^2y$ at $(3,2)$.

find $\frac{\partial f}{\partial x}$ at $(3,2)$ $f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0^3 + h, y_0^2) - f(x_0, y_0)}{h}$

1. $f(3+h, 2) \to$ plug $x = (3+h)$ $y = 2$ in $f(x,y)$

$(3+h)^2 = a^2 + 2ab + b^2$ ANS $-61 - 50h - 8h^2$

2. $f(3,2) \to$ plug $x = 3$, $y = 2$ in $f(x,y)$ ANS -61.
3. apply $f_x(x_0,y_0)$ formula ANS -50.

find $\frac{\partial f}{\partial y}$ at $(3,2)$ $f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0^3, y_0^2 + h) - f(x_0, y_0)}{h}$

1. $f(3, 2+h) \to$ plug $x = 3$ $y = (2+h)$ in $f(x,y)$ ANS $-61 - 31h$
2. apply $f_y(x_0,y_0)$ formula ANS -31.

find the slope of the tangent line to $f(x,y)$ at P. lying in plane $x = 1$ $y = 1$.

1. take $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}$ treat x as constant
2. evaluate result at $x = 1, y = 1$ $4y^3 \big|_{(1,1)}$
3. take $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}$ treat y as constant
4. evaluate result at $x = 1, y = 1$ $3x^2 \big|_{(1,1)}$

For the functions $w = 9x^2 + 3y^2$, $x = \cos t$, and $y = \sin t$, express $\frac{dw}{dt}$ as a function of t, by using the chain rule and by expressing w in terms of t and differentiating directly with respect to t. Then, evaluate $\frac{dw}{dt}$ at $t = \frac{\pi}{2}$.

Express $\frac{dw}{dt}$ as a function of t.

Partial differentiate w with respect to x treat y as a constant $\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}$

Partial differentiate w with respect to y treat x as a constant $\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}$

differentiate $x = \cos t$ with respect to t $\frac{dx}{dt} = \frac{d}{dt}(\cos t)$

differentiate $y = \sin t$ with respect to t $\frac{dy}{dt} = \frac{d}{dt}(\sin t)$

write $\frac{dw}{dt}$ using the chain rule $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

Plug $x = \cos t$, $y = \sin t$ after simplifying

Evaluate $\frac{dw}{dt}$ at $t = \frac{\pi}{2}$. $\frac{dw}{dt}\Big|_{t = \frac{\pi}{2}}$

consider the functions $z = -3e^x \ln y$, $x = \ln(u \cos v)$, and $y = u \sin v$.

a) express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as functions of both u and v by using the chain rule and by expressing z directly in terms of u and v before differentiating.

b) evaluate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at $(u,v) = \left(3, \frac{\pi}{3}\right)$

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$ Partial differentiate z with respect to x $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}$

Partial differentiate x with respect to u $\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}$ $\frac{\partial}{\partial x}(\ln x) = \frac{1}{x}$

may need chain rule afuera. adentro

$f'(g(x)) = f'(g(x))\,g'(x)$ Partial differentiate z with respect to y $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}$

Partial differentiate y with respect to u $\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}$

cont. apply formula, insert values of x and y into equation, factor. $e^{\ln x} = \ln$

b) evaluate $\partial z/\partial u$ at $(u,v) = (3, \pi/3)$ $\frac{\partial z}{\partial u}\Big|_{(3,\pi/3)}$

repeat all steps but dz/dx and dz/dy since you already have those values. $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$

evaluate $\frac{\partial z}{\partial v}$ at $(u,v) = \left(3, \frac{\pi}{3}\right)$ $\frac{\partial z}{\partial v}\Big|_{(3,\pi/3)}$

assuming $4x^3 - y^2 - 3xy = 0$ defines y as a differentiable function of x, use the theorem $\frac{dy}{dx} = -\frac{F_x}{F_y}$ to find $\frac{dy}{dx}$ at the point (1,1).

differentiate $F(x,y)$ with respect to x.

$F_x = \frac{\partial}{\partial x}$ treat y as a constant

differentiate $F(x,y)$ with respect to y.

$F_y = \frac{\partial}{\partial y}$ treat x as a constant

write $\frac{dy}{dx}$ using the theorem $\frac{dy}{dx} = -\frac{F_x}{F_y}$ $\frac{dy}{dx}\Big|_{(1,1)}$

substitute $x = 1$ and $y = 1$ and evaluate at the given point.

Find $\frac{dw}{dr}$ when $r = 2$ and $s = -2$ if $w = (x + y + z)^2$

$x = r - s$, $y = \cos(r+s)$, $z = \sin(r+s)$

$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$ find all derivatives then plug into formula.

solve for $x = r - s$, $y = \cos(r+s)$, $z = \sin(r+s)$ by plugging values of r and s.

go back to formula and plug x,y,z values. $\frac{\partial w}{\partial r}\Big|_{s=-2}^{r=2}$

assume that $w = f(s^3 + t^2)$ and $f'(x) = e^x$.

Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$

differentiate w with respect to t $\frac{\partial w}{\partial t}$ may need chain rule

substitute $x = (s^3 + t^2)$ in $f'(x) = e^x$

substitute $f'(s^3 + t^2) = e^{s^3 + t^2}$ in $2t f'(s^3 + t^2)$

assume that $z = f(w)$ $w = g(x,y)$

$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$ take derivative of $\frac{\partial x}{\partial r}$ take derivative of $\frac{\partial y}{\partial r}$

g_x g_y plug g_x g_y.

$\frac{\partial z}{\partial r} = f'(w) \cdot w$ then plug $r = 5$, $s = 0$

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$ take all steps.

Find the gradient of the function $f(x,y)$ at P.

$\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ ⊥ to surface $f(x,y)$

take partials, then plug point. $\Delta f\big|_{x,y}$ $i + j + k$

derivative of function P_0 in the direction of A.

take partials, then plug point. $\Delta f\big|_{x,y}$ gradient $i + j + k$

you have vector A. Find $|A|$ then use formula $u = \frac{A}{|A|}$, u_x "i" and u_y "j"

$D_A f = \frac{\partial f}{\partial x} u_x + \frac{\partial f}{\partial y} u_y$

$\frac{d}{dx}\left[\frac{f(x)}{f(y)}\right] = \frac{g(x) f'(x) - f(x) g'(x)}{[g(x)]^2}$

Find the direction in which func. increases and decreases at P.

take partials, then plug point. $\Delta f\big|_{x,y}$ gradient $i + j + k$

you have vector Δf. Find $|\Delta f|$

then formula $u = \frac{\Delta f(x,y)}{|\Delta f|_{(x,y)}} = i + j + k$. $+ = u$ $- = -u$

tangent line equation

$f_x(x_0)(x - x_0) + f_y(y_0)(y - y_0) = 0$ from gradient vector

Find parametric eq. for line perpendicular to the graph of eq. $5x^2 + 5y^2 + 5z^2 = 105$ at the point $(4,-2,-1)$.

1. Find gradient $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$
2. evaluate at P. $\nabla f\big|_{(4,-2,-1)}$ $i + j + k$
3. Find parametric eq. plug $(x_0, y_0, z_0) = (4,-2,-1)$ into $x = x_0 + 40t$ $y = y_0 - 20t$ $z = z_0 - 10t$.
4. eq. is $r(t) = (4 + 40t)i + (-2 - 20t)j + (-1 - 10t)k$

Find the equation for the tangent plane and the normal line at $P(1,3,3)$ on f_x.

The linearization of a function f(x,y) at a point (x_0,y_0), where f is differentiable,

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

1. Find $F(x_0, y_0)$ = plug point into eq. 2. Find partial derivatives 3. Plug in point into derivative
4. Plug ans. into formula. $L(x,y) = 10x + 10y - 49$

14.7 Extreme values and Saddle points

Local Extrema — 2nd derivative test.

$f_{xx}<0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ then $F(a,b)\rightarrow$ local max
$f_{xx}>0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ then $F(a,b)\rightarrow$ local min
$f_{xx}f_{yy}-f_{xy}^2<0$ then $F(a,b)\rightarrow$ saddle point
$f_{xx}f_{yy}-f_{xy}^2=0$ then $F(a,b)\rightarrow$ inconclusive

1. Find Fx and Fy
2. Find point of critical point when $x=0$ and $y=0$. (solve y, substitute in x)
3. Find Fxx, Fyy, Fxy
4. Find D (discriminant)
5. Evaluate critical point of $F(x,y)$ to find value.

$\boxed{L1 \quad y=0 \quad 0\le x\le 4 \quad F(x,0)=x^2-2(x)+4(0) = x^2 \quad F(0)=0, \; F(4)=16}$

Absolute Extrema

1. Find Fx and Fy
2. Find point of critical point when $x=0$ and $y=0$. (solve y, substitute in x)
3. Sketch and mark segments with endpoints (L1 (0,2),(2,0))
4. table $x|y|F$ to record values and find abs. max and abs. min.
   $x^2-6x+12$ Quadratic expression! Find min value
   take derivative, set to zero, solve for x.
   evaluate at found point.
5. Evaluate critical point plug into original expression.

EXAM 3

Reverse the order of integration

Imagine a vertical line passing through the graph from bottom to top. Identify the first y value it encounters and then the last. These are the new y-limits of integration.

Next identify the limits on x. Identify the lowest and then the highest value x can take. These are the new x-limits of integration.

$$\int_0^{\frac{1}{2}}\int_0^{x^3}\cos(8\pi x^4)\,dy\,dx \qquad \frac{1}{32\pi}\int_0^{\frac{\pi}{2}}\cos(u)\,du$$

$u=8\pi x^4$, $du=32\pi x^3\,dx$, $dx=\frac{1}{32\pi x^3}du$

$$\int\left[y\cos(8\pi x^4)\right]_0^{x^3}dx \qquad \frac{1}{32\pi}\left[\sin(u)\right]_0^{\frac{\pi}{2}}$$

$$\int\left[x^3\cos(8\pi x^4)\right]dx \qquad \frac{1}{32\pi}\left(\sin\tfrac{\pi}{2}-\sin 0\right)$$

Adjust boundaries: $x=0 \rightarrow u=0$, $x=\frac{1}{2}\rightarrow u=\frac{\pi}{2}$

$$\frac{1}{32\pi}(1-0)$$

$$\int_0^{\frac{\pi}{2}}\cancel{x^3}\cos(u)\frac{1}{32\pi \cancel{x^3}}du \qquad \frac{1}{32\pi}(1) = \frac{1}{32\pi}$$

$$\int_0^{\frac{\pi}{2}}\frac{\cos(u)}{32\pi}du$$

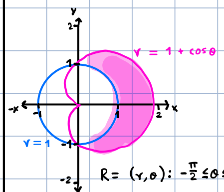
Find the area of: Cardioid $r=1+\cos\theta$ and a circle $r=1$.

$$A=\iint_R dA$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_1^{1+\cos\theta} r\,dr\,d\theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{1}{2}r^2\right]_1^{1+\cos\theta}d\theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{1}{2}(1+\cos\theta)^2-\frac{1}{2}\right]d\theta$$



$R=(r,\theta): -\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\; 1\le r\le 1+\cos\theta$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta+\frac{1}{2}\cos^2\theta\,d\theta \qquad x^2+2ax+a^2=(x+a)^2$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta+\frac{\cos^2\theta}{2}\,d\theta \qquad \cos^2\theta=\frac{1+\cos 2\theta}{2}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta+\frac{1}{4}(1+\cos 2\theta)\,d\theta$$

$$\left[\sin\theta+\frac{1}{4}\left(\theta+\frac{1}{2}\sin 2\theta\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$\left[\sin\tfrac{\pi}{2}+\tfrac{1}{4}\left(\tfrac{\pi}{2}+\tfrac{1}{2}\sin\pi\right)-\sin\tfrac{-\pi}{2}+\tfrac{1}{4}\left(\tfrac{-\pi}{2}+\tfrac{1}{2}\sin{-\pi}\right)\right]$$

$$\left(1+\frac{\pi}{8}\right)-\left(-1-\frac{\pi}{8}\right)=2+\frac{\pi}{4}$$

14.8 Lagrange Multipliers

Finding Extrema subject to a constraint

1. Find $Fx=\lambda gx$ $Fy=\lambda gy$ $Fz=\lambda gz$ Simultaneously!
2. Plug x,y,z values into constraint function $g(x,y,z)$.
3. Solve for λ.
4. Plug λ into x,y,z results from step 1 to get points.
5. Evaluate $F(x,y,z)$ at given points from step 4.

15.3 Area by Double Integration

Given a curve and a line. Sketch. Use vertical/horizontal line test to find lower/upper x- and y-limits. Setup integral.

$$A=\int_R\int dA$$

15.3 Average Value

Find the area of the region R.

$A(R)=\frac{\pi}{3}\cdot\frac{7\pi}{6}=\frac{7\pi^2}{18}$ $\qquad f_{avg}=\frac{1}{\text{area of }R}\iint_R f\,dA$

$$\frac{1}{\frac{7\pi^2}{18}}\int_0^{\frac{\pi}{3}}\int_0^{\frac{7\pi}{6}}\sin(x+y)\,dy\,dx$$

$$\frac{18}{7\pi^2}\int_0^{\frac{\pi}{3}}\left[-\cos(x+y)\right]_0^{\frac{7\pi}{6}}dx$$

$$\frac{18}{7\pi^2}\int_0^{\frac{\pi}{3}}\left[-\cos\left(x+\tfrac{7\pi}{6}\right)-\left(-\cos(x+0)\right)\right]dx$$

$$\frac{18}{7\pi^2}\int_0^{\frac{\pi}{3}}\left(\cos x-\cos\left(x+\tfrac{7\pi}{6}\right)\right)dx$$

$$\frac{18}{7\pi^2}\left[\sin x\right]_0^{\frac{\pi}{3}}-\left[\sin\left(x+\tfrac{7\pi}{6}\right)\right]_0^{\frac{\pi}{3}}$$

$$\frac{18}{7\pi^2}\left[\sin\tfrac{\pi}{3}-\sin 0\right]-\left[\sin\left(\tfrac{\pi}{3}+\tfrac{7\pi}{6}\right)-\sin\left(0+\tfrac{7\pi}{6}\right)\right]$$

$$\frac{18}{7\pi^2}\left(\tfrac{\sqrt{3}}{2}\right)-\left(\sin\left(\tfrac{3}{2}\right)-\sin\left(\tfrac{7\pi}{6}\right)\right)$$

$$\frac{18}{7\pi^2}\left(\tfrac{\sqrt{3}}{2}\right)-\left(-1-\left(-\tfrac{1}{2}\right)\right)$$

$$\frac{18}{7\pi^2}\left(\tfrac{\sqrt{3}}{2}\right)-\left(-\tfrac{1}{2}\right)$$

$$\frac{18}{7\pi^2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)$$

$$\frac{18}{7\pi^2}\left(\frac{\sqrt{3}+1}{2}\right)$$

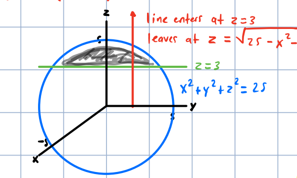
$$\frac{9(\sqrt{3}+1)}{7\pi^2}$$

15.5 Triple Integrals in Rectangular Coordinates

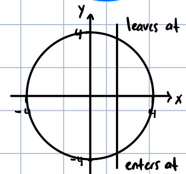
$$V=\iiint_D dV \qquad \text{Average value}=\frac{1}{\text{volume of }D}\iiint_D F(x,y,z)\,dV$$

Finding limits of integration (in dz dy dx order)

$x^2+y^2+z^2=25$
$z^2=25-x^2-y^2$
$z=\sqrt{25-x^2-y^2}$



$f_1(x,y)=3$
$F_2(x,y)=\sqrt{25-x^2-y^2}$
$x^2+y^2+3^2=25$
$x^2+y^2=25-9=16$



$x^2+y^2=16$
$y^2=16-x^2$
$y=\pm\sqrt{16-x^2}$

$-\sqrt{16-x^2}\le y\le\sqrt{16-x^2}$

$-4\le x\le 4$

So, $$\iiint_D F(x,y,z)\,dV=\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_3^{\sqrt{25-x^2-y^2}}F(x,y,z)\,dz\,dy\,dx$$

15.1 Double and iterated integrals over rectangles.

Fubini's Theorem for integral over region R.

$$R=\{a\le x\le b, c\le y\le d\}$$

$$\iint_R f(x,y)\,dA=\int_c^d\int_a^b f(x,y)\,dx\,dy=\int_a^b\int_c^d f(x,y)\,dy\,dx$$

$$\int_0^1\int_0^{10}\frac{yx^4}{y^2+1}dx\,dy \qquad R:\; 0\le x\le 10,\quad 0\le y\le 1$$

treating y as a constant

$$\frac{y}{y^2+1}\int_0^{10}x^4\,dx$$

$$\frac{y}{y^2+1}\cdot\left.\frac{x^5}{5}\right|_0^{10}$$

$$\frac{y}{y^2+1}\left(\frac{10^5}{5}-\frac{0^5}{5}\right)$$

$$\frac{y}{y^2+1}(2000)$$

$$\frac{2000y}{y^2+1}$$

$$\int_0^1\frac{2000y}{y^2+1}dy \qquad \text{let } u=y^2+1, \quad du=2y\,dy, \quad dy=\frac{1}{2y}du$$

$$2000\int_0^1\frac{y}{y^2+1}dy$$

$$2000\int_1^2\frac{\cancel{y}}{u}\frac{1}{2\cancel{y}}du \qquad \text{Adjust integral boundaries}$$

$y=0\rightarrow u=1$
$y=1\rightarrow u=2$

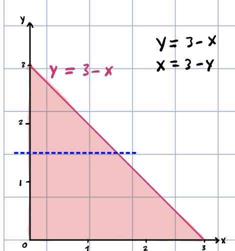
$$2000\int_1^2\frac{1}{2u}du$$

$$2000\cdot\frac{1}{2}\int_1^2\frac{1}{u}du = 1000\left(\ln 2-\cancel{\ln 1}\right) \qquad \ln 1=0$$

$$2000\cdot\frac{1}{2}\left[\ln u\right]_1^2 = 1000\ln 2$$

Double Integrals as volumes

$$V=\int_R\int f(x,y)\,dA$$



$y=3-x$
$x=3-y$

$$\int_0^3\int_0^{3-y}(45-5y^2)\,dx\,dy$$

$$\int_0^3\left[45x-5y^2x\right]_0^{3-y}dy$$

$$\int_0^3\left[45(3-y)-5y^2(3-y)\right]dy$$

$$\int_0^3\left[135-45y-15y^2+5y^3\right]dy$$

$$\int_0^3\left(5y^3-15y^2-45y+135\right)dy$$

$$\left[\frac{5}{4}y^4-5y^3-\frac{45}{2}y^2+135y\right]_0^3=\frac{675}{4}$$

15.4 Double Integrals in Polar Form

Change a Cartesian integral into an equivalent polar integral.

$$\iint_R f(x,y)\,dx\,dy=\iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$x=r\cos\theta$, $y=r\sin\theta$, $r^2=x^2+y^2$, $\tan\theta=\frac{y}{x}$, $dx\,dy=r\,dr\,d\theta$

$$\int_0^{\ln 3}\int_0^{\sqrt{(\ln 3)^2-y^2}}e^{\sqrt{x^2+y^2}}dx\,dy \qquad 0\le y\le\ln 3 \;\text{ and }\; 0\le x\le\sqrt{(\ln 3)^2-y^2}$$

Use polar coordinates:

$x=r\cos\theta$, $y=r\sin\theta$

So, $x^2+y^2=r^2$ and $dx\,dy=r\,dr\,d\theta$

take $x^2=\sqrt{(\ln 3)^2-y^2}$
$x^2=(\ln 3)^2-y^2$
$x^2+y^2=(\ln 3)^2$

this is a circle of radius $\ln 3$.

$D=\; 0\le r\le\ln 3,\quad 0\le\theta\le\frac{\pi}{2}$

$$\int_0^{\ln 3}\int_0^{\sqrt{(\ln 3)^2-y^2}}e^{\sqrt{x^2+y^2}}dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^{\ln 3}e^r r\,dr\,d\theta$$

evaluate the integral — Integration by parts

$$\int u\,v'=uv-\int u'v$$

$u=r$, $v'=e^r$
$u'=1\,dr$, $v=\int e^r=e^r$

$$\int_0^{\frac{\pi}{2}}\int_0^{\ln 3}e^r r\,dr\,d\theta$$

$$\left[e^r r-\int e^r\,dr\right]_0^{\ln 3}$$

$$\int_0^{\frac{\pi}{2}}\left[e^r r-e^r\right]_0^{\ln 3}d\theta$$

$$\int_0^{\frac{\pi}{2}}\left(e^{\ln 3}\ln 3-e^{\ln 3}\right)-\left(e^0\cdot 0-e^0\right)d\theta$$

$$\int_0^{\frac{\pi}{2}}(3\ln 3-3+1)\,d\theta$$

$$\int_0^{\frac{\pi}{2}}(3\ln 3-2)\,d\theta$$

$$\left[3\theta\ln 3-2\theta\right]_0^{\frac{\pi}{2}}$$

$$\left[3\tfrac{\pi}{2}\ln 3-2\tfrac{\pi}{2}\right]_0^{\frac{\pi}{2}}$$

$$\frac{3\pi}{2}\ln 3-\pi$$

Find average value

$$\text{Average value}=\frac{1}{\text{volume of }D}\iiint_D F(x,y,z)\,dV$$

Volume of rectangular solid: $x\cdot y\cdot z$.

$$\frac{1}{9}\int_0^1\int_0^3\int_0^3(x^2+y^2+z^2)\,dx\,dy\,dz \qquad \frac{1}{9}\int_0^1(27+27+9z^2)\,dz$$

$$\frac{1}{9}\int_0^1\int_0^3\left[\frac{x^3}{3}+xy^2+xz^2\right]_0^3dy\,dz \qquad \frac{1}{9}\int_0^1(54+9z^2)\,dz$$

$$\frac{1}{9}\int_0^1\int_0^3\left[\frac{3^3}{3}+3y^2+3z^2\right]_0^3dy\,dz \qquad \frac{1}{9}\left[54z+3z^3\right]_0^1$$

$$\frac{1}{9}\int_0^1\int_0^3(9+3y^2+3z^2)\,dy\,dz \qquad \frac{1}{9}\left[54_1+3_1^3\right]_0^1$$

$$\frac{1}{9}\int_0^1\left[9y+y^3+3yz^2\right]_0^3dz \qquad \frac{1}{9}(57)$$

$$\frac{1}{9}\int_0^1\left[9_3+3^3+3_3z^2\right]_0^3dz \qquad \frac{19}{3}$$

15.8 Substitutions in Multiple Integrals

Use the transformation $u=4x+3y$, $v=x+3y$ to evaluate the given integral for the region R bounded by the lines $y=-\frac{4}{3}x+1$, $y=-\frac{4}{3}x+4$, $y=-\frac{1}{3}x$ and $y=-\frac{1}{3}x+2$.

$$\iint_R f(x,y)\,dx\,dy=\iint_G f(g(u,v),h(u,v))\,|J(u,v)|\,du\,dv$$

$$J(u,v)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

$$\iint_R UV\,|J(u,v)|\,du\,dv$$

$$\iint_R\left(\underset{\text{write}}{4x^2+15xy+9y^2}\right)dx\,dy=\iint_R\underbrace{(4x+3y)}_{u}\underbrace{(x+3y)}_{v}\underbrace{dx\,dy}_{du\,dv}$$

since $U=4x+3y$, $V=x+3y$, solve for x and y.

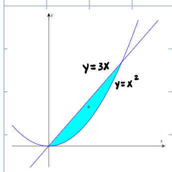
$V=x+3y$ $\qquad U=4x+3y \qquad x=V-3y$
$x=V-3y$ $\qquad U=4(V-3y)+3y \qquad x=V-3\left(\frac{4V-U}{9}\right)$
$U=4V-12y+3y \qquad x=V-\frac{4V+U}{3}$
$U=4V-9y \qquad x=\frac{U-V}{3}$
$9y=4V-U$
$y=\frac{4V-U}{9}$

$$|J(u,v)|=\begin{vmatrix}\frac{\partial x}{\partial u}\left[\frac{U-V}{3}\right] & \frac{\partial x}{\partial v}\left[\frac{U-V}{3}\right]\\ \frac{\partial y}{\partial u}\left[\frac{4V-U}{9}\right] & \frac{\partial y}{\partial v}\left[\frac{4V-U}{9}\right]\end{vmatrix}=\begin{vmatrix}\frac{1}{3} & -\frac{1}{3}\\ -\frac{1}{9} & \frac{4}{9}\end{vmatrix}$$

$$=\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)-\left(-\frac{1}{9}\right)\left(-\frac{1}{3}\right)=\frac{1}{9}$$

15.2 Double integrals over general regions of R^2

$$R=\{a\le x\le b,\; g(x)\le y\le h(x)\} \qquad \iint_R f(x,y)\,dA=\int_a^b\int_{g(x)}^{h(x)}f(x,y)\,dy\,dx$$



1. Solve y equations to find point of intersection of both curves.

$y=3x$
$y=x^2$
$3x=x^2$
$x^2-3x=0$
$x(x-3)=0$
$0\le x\le 3$

$$\iint_R F(x,y)\,dA=\int_0^3\int_{x^2}^{3x}dy\,dx \quad \text{Vertical}$$

$$R=\{c\le y\le d,\; g(y)\le x\le h(y)\} \qquad \iint_R f(x,y)\,dA=\int_c^d\int_{g(y)}^{h(y)}f(x,y)\,dx\,dy$$

$y=3x$ $\quad y=x^2$ $\quad y=3(3)$ $\quad y=(3)^2=9$
$x=\frac{y}{3}$ $\quad x=\sqrt{y}$ $\quad y=3(0)$ $\quad y=(0)^2=0$
$\frac{y}{3}\le x\le\sqrt{y}$

$$\iint_R F(x,y)\,dA=\int_0^9\int_{\frac{y}{3}}^{\sqrt{y}}dx\,dy \quad \text{Horizontal}$$

Change a polar integral into a Cartesian integral.

$x=r\cos\theta$, $r^2=x^2+y^2$, $\tan\theta=\frac{y}{x}$, $dx\,dy=r\,dr\,d\theta$
$y=r\sin\theta$,

$$\int_0^{\frac{\pi}{4}}\int_0^{5\sec\theta}r^5\sin^2\theta\,dr\,d\theta$$

since the limits of r are $r=0$ and $r=5\sec\theta$

$r=5\sec\theta$ $\qquad \sec\theta=\frac{1}{\cos\theta}$

$r=\frac{5}{\cos\theta}$

$r\cos\theta=5$

$x=5$

Rewrite:

$$\int_0^{\frac{\pi}{4}}\int_0^{5\sec\theta}r^4\sin^2\theta\; r\,dr\,d\theta$$

$$\int_0^{\frac{\pi}{4}}\int_0^{5\sec\theta}r^2\cdot r^2\sin^2\theta\; r\,dr\,d\theta$$

since the limits of θ are: $\theta=0$ and $\theta=\frac{\pi}{4}$

$\theta=\frac{\pi}{4}$ $\qquad \frac{y}{x}=\tan\frac{\pi}{4}$

$\tan\theta=\frac{y}{x}$ $\qquad \frac{y}{x}=1.\quad y=x$

$$\int_0^{\frac{\pi}{4}}\int_0^{5\sec\theta}r^2(r\sin\theta)^2\,r\,dr\,d\theta$$

R: $0\le x\le 5$, $0\le y\le x$

$$\int_0^{\frac{\pi}{4}}\int_0^{5\sec\theta}r^2(r\sin\theta)^2\,r\,dr\,d\theta=\int_0^5\int_0^x(x^2+y^2)(y)^2\,dy\,dx$$

therefore,

$$\iint_R (4x+3y)(x+3y)\,dx\,dy = \iint_R uv\,|J(u,v)|\,du\,dv = \frac{1}{9}\iint_G uv\,du\,dv$$

transform the equation $y = -\frac{4}{3}x + 1$

$$\frac{4v-u}{9} = -\frac{4}{3}\left(\frac{u-v}{3}\right) + 1$$
$$9\cdot\frac{4v-u}{9} = -\frac{4u+4v}{9} + 1\cdot 9$$
$$\cancel{4v} - u = -4u + \cancel{4v} + 9$$
$$-9 = -4u + u$$
$$-9 = -3u$$
$$u = 3$$

transform the equation $y = -\frac{4}{3}x + 4$

$$\frac{4v-u}{9} = -\frac{4}{3}\left(\frac{u-v}{3}\right) + 4$$
$$9\cdot\frac{4v-u}{9} = -\frac{4u+4v}{9} + 4\cdot 9$$
$$\cancel{4v} - u = -4u + \cancel{4v} + 36$$
$$-36 = -4u + u$$
$$-36 = -3u$$
$$u = 12$$

transform the equation $y = -\frac{1}{3}x$

$$\frac{4v-u}{9} = -\frac{1}{3}\left(\frac{u-v}{3}\right)$$
$$9\cdot\frac{4v-u}{9} = -\frac{u+v}{9}\cdot 9$$
$$4v - \cancel{u} = -\cancel{u} + v$$
$$3v = 0$$
$$v = 0$$

transform the equation $y = -\frac{1}{3}x + 2$

$$\frac{4v-u}{9} = -\frac{1}{3}\left(\frac{u-v}{3}\right) + 2$$
$$9\cdot\frac{4v-u}{9} = -\frac{u+v}{9} + 2\cdot 9$$
$$4v - \cancel{u} = -\cancel{u} + v + 18$$
$$3v = 18$$
$$v = 6$$

EXAM 3

Hence,

$$\frac{1}{9}\iint_G uv\,du\,dv \qquad \frac{1}{9}\int_0^6 \left[\frac{u^2}{2}\right]_3^{12} v\,dv \qquad \frac{15}{2}\left[\frac{v^2}{2}\right]_0^6$$

$$\frac{1}{9}\int_0^6\int_3^{12} uv\,du\,dv \qquad \frac{1}{9}\int_0^6 \left(\frac{135}{2}\right) v\,dv \qquad \frac{15}{2}\left[\frac{v^2}{2}\right]_0^6$$

$$\frac{15}{2}(18) = 135$$

16.1 Line Integrals of Scalar Functions

$\int_C (x+y)\,ds$ over the straight-line segment $x=3t,\ y=(6-3t),\ z=0$ from $(0,6,0)$ to $(6,0,0)$

when
$x = 3t = 0 \quad t=0$
$3t = 6 \quad t=2$

$r(t) \to 3t\,i + (6-3t)j + 0k$
$r'(t) \to 3i + 3j$
$|r'(t)| = \sqrt{9+9} = 3\sqrt{2} \to |V(t)|$

so, $0 \le t \le 2$

so, $\int_C (x+y)\,ds = \int_0^2 (x+y)\,|V(t)|\,dt \qquad ds = |V(t)|$

$$\int_0^2 (\cancel{3t} + 6 - \cancel{3t})\cdot 3\sqrt{2}\,dt = \int_0^2 18\sqrt{2}\,dt = 18\sqrt{2}\,[t]_0^2 = 18\sqrt{2}\,(2-0) = 36\sqrt{2}$$

Integrating over a curve

$\int_C (xy + x + z)\,ds$ along the curve $r(t) = 2t\,i + t\,j + (8-2t)k,\ 0 \le t \le 1.$

$r'(t) \to 2i + j - 2k$
$|r'(t)| = \sqrt{4+1+4} = 3dt \to |V(t)| = ds$

$$\int_C f(x,y,z)\,ds = \int_a^b f(g(t), h(t), k(t))\,|v(t)|\,dt$$

$$\int_C (xy+x+z)\,ds = \int_0^1 (2t\cdot t + \cancel{2t} + (8 - \cancel{2t}))\,3dt = \int_0^1 3(2t^2 + 8)\,dt$$

$$3\left[\frac{2}{3}t^3 + 8t\right]_0^1 = 26$$

Integrate $f(x,y,z) = x + \sqrt{y} - z^4$ over the path from (0,0,0) to (1,1,1) given by
C_1: $r(t) = t\,i + t^2 j,\ 0 \le t \le 1$
C_2: $r(t) = i + j + tk,\ 0 \le t \le 1.$

use the equation

$$\int_C (x + \sqrt{y} - z^4)\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$$

for C_1: $r(t) = t\,i + t^2 j,\quad 0 \le t \le 1$
$x(t) = t,\ y(t) = t^2,\ z(t) = 0$
$x'(t) = 1,\ y'(t) = 2t,\ z'(t) = 0$

$ds = |v'(t)|$
$= \sqrt{1^2 + (2t)^2 + 0^2}\,dt$
$\sqrt{1 + 4t^2}\,dt$

from starting / ending point

from parametrization

$$\int_C (x + \sqrt{y} - z^4)\,ds = \int_0^1 (t + t - 0^4)\sqrt{1+4t^2}\,dt$$

$$\int_0^1 2t\sqrt{1+4t^2}\,dt \qquad u = 1 + 4t^2 \quad du = 8t\,dt \quad t\,dt = \frac{1}{8}du$$

$$\therefore \int t\frac{\sqrt{u}}{8}\,du = \int \frac{\sqrt{u}}{8}\,du = \frac{u^{\frac{1}{2}+1}}{8\cdot\frac{1}{2}+1} = \frac{u^{\frac{3}{2}}}{12} \qquad \frac{(1+4t^2)^{3/2}}{12}$$

$$2\left[\frac{(1+4t^2)^{3/2}}{12}\right]$$

$$\frac{1}{6}\left[(1+4t^2)^{3/2}\right]_0^1 = \frac{1}{6}(5)^{\frac{3}{2}} - 1 = \frac{1}{6}(5\sqrt{5} - 1) \quad ①$$

for C_2: $r(t) = i + j + tk,\quad 0 \le t \le 1$
$x(t) = 1,\ y(t) = 1,\ z(t) = t$
$x'(t) = 0,\ y'(t) = 0,\ z'(t) = 1$

$ds = |v'(t)|$
$= \sqrt{1^2}\,dt$
$= 1\,dt$

$$\int_C (x + \sqrt{y} - z^4)\,ds = \int_0^1 (1 + 1 - t^4)\,dt = \int_0^1 (2 - t^4)\,dt$$

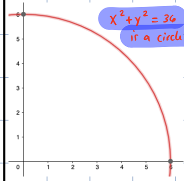
$$(2 - t^4)\,dt = \left[2t - \frac{t^5}{5}\right]_0^1 = \frac{9}{5} \quad ②$$

thus, $\int_C (x + \sqrt{y} - z^4)\,ds$

$$= \frac{1}{6}(5\sqrt{5} - 1) + \frac{9}{5} = \frac{5\sqrt{5}}{6} - \frac{1}{6} + \frac{9}{5} = \frac{5\sqrt{5}}{6} + \frac{49}{30}$$

ANS.

Integrate $f(x,y) = x + y$ over the curve C: $x^2 + y^2 = 36$ in the first quadrant from $(6,0)$ to $(0,6)$.



$x^2 + y^2 = 36 \quad 0 \le t \le \frac{\pi}{2}$

$r(t) = (6\cos t)i + (6\sin t)j$
$v(t) = (-6\sin t)i + (6\cos t)j$
$|v(t)| = \sqrt{(-6\sin t)^2 + (6\cos t)^2}$
$= \sqrt{36\sin^2 t + 36\cos^2 t}$
$= \sqrt{36(\sin^2 t + \cos^2 t)}$
$= 6\,dt$

$$\int_C f(x,y)\,ds = \int_0^{\frac{\pi}{2}} f(6\cos t, 6\sin t)\cdot 6\,dt$$

$$\int_0^{\frac{\pi}{2}} (6\cos t + 6\sin t)\cdot 6\,dt$$

$$\int_0^{\frac{\pi}{2}} 6(\cos t + \sin t)\cdot 6\,dt$$

$$36\int_0^{\frac{\pi}{2}} (\cos t + \sin t)\,dt$$

$$36\left[\sin t - \cos t\right]_0^{\frac{\pi}{2}}$$

$$36\left[\sin\frac{\pi}{2} - \cos\frac{\pi}{2} - (\sin 0 - \cos 0)\right]$$

$$36\left[1 - 0 - (0 - 1)\right] = 36\cdot 2 = 72$$

Find the mass of a wire that lies along the curve $r(t) = (t^2 - 5)j + 2t\,k \quad 0 \le t \le 3$, if the density is $\delta = \frac{3}{2}t$.

$r(t) \to (t^2 - 5)j + 2t\,k$
$r'(t) \to 2t\,j + 2k$
$|r'(t)| = \sqrt{4t^2 + 4} = 2\sqrt{t^2+1}\,dt \to |V(t)|$

$$M = \int_C \delta(x,y,z)\,ds = \int_C \delta(x,y,z)\,|v(t)|\,dt$$

$$M = \int_0^3 \frac{3}{\cancel{2}}t\cdot\cancel{2}\sqrt{t^2+1}\,dt = \int_0^3 3t\sqrt{t^2+1}\,dt$$

$u\ (t^2+1)^{\frac{1}{2}}$
$du\ \frac{1}{2}(t^2+1)^{-\frac{1}{2}}\cdot 2t$
$du\ \frac{\cancel{2}t}{\cancel{2}(t^2+1)^{\frac{1}{2}}} = \frac{t}{\sqrt{t^2+1}}$

$t = 3 \quad u = \sqrt{10}$
$t = 1 \quad u = 1$
$t = \sqrt{t^2+1}\,du$
$t = u\,du$

$$= 3\int_1^{\sqrt{10}} u\cdot u\,du = 3\int_1^{\sqrt{10}} u^2\,du$$

$$= \left[\frac{u^3}{3}\right] = \frac{(\sqrt{10})^3}{3} - \frac{1}{3} = \frac{10^{\frac{1}{2}\cdot 3}}{3} - \frac{1}{3}$$

$$= \frac{10^{\frac{1}{2}\cdot 3}}{3} - \frac{1}{3} = \frac{10^{\frac{3}{2}}}{3} - \frac{1}{3} = \frac{10\sqrt{10} - 1}{\cancel{3}}\cdot\cancel{3}$$

$$= 10\sqrt{10} - 1$$